A Regime Switching Model under the Heston Stochastic Volatility

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I. Introduction

In this report I detail the background information, methods, and results of my summer research done through the Center for Applied Mathematics at the University of St. Thomas. In collaboration with Dr. Tang from the Actuarial Science department, we investigated building a regime switching options pricing model under the Heston stochastic volatility framework. This project would build upon previous work done with incorporating regime switching into Heston type models. The model itself aims to more accurately represent stock market prices and their movement. From it we can derive theoretical prices of options, a tool widely used by insurance companies to manage risk and compare those prices to actual market prices and those obtained from other models. Once we have built a functioning model, we plan to implement price surfaces to analyze.

II. Background Overview

First, before we go any further into options pricing, there are some terms that are important to understand.

The most important concept to firmly understand is that of an option. What is an option? An option is a contract that one can buy that gives them the right to buy or sell the underlying asset at the determined price, called the strike price. There are two types of options: a call and a put. A call gives the owner the right to buy at the strike price. A put gives the owner the right to sell at the strike price. For example, one could buy an option for Apple stock. If Apple stock is at $100 right now and they thought that Apple stock was going to increase, they could buy a call option for $110. If the underlying asset (Apple stock) did increase to $115, then the owner now has the right to buy apple stock at $110 and sell it at $115. An important point to this process is that you do not have to exercise the option. Therefore, if Apple stock did not actually increase but instead decreased, you could choose to let the option expire.
The applications of options can go much further beyond just arbitrage as there are countless strategies traders can employ with them. Such strategies like a covered call, married put, or butterfly spreads to name a few. What all these strategies have in common are that they are trying to reduce the amount of risk the owner has. Financial risk is simply the possibility that an individual, investor, or company might lose money. One such example is that just holding options themselves instead of the stock as a much cheaper and safer way to “own” a stock. This is because you are just buying the right to them and, in the end, do not have to go forward. Since making money is the primary goal of any company, it is no surprise that managing risk with options has become extremely widespread.

Furthermore, here are a few more concepts that relate to our paper: a regime is a period of the stock market that is characterized by a certain return rate or volatility level. These regimes are usually longstanding and are dominated by a single characteristic which most often is determined by a country’s current monetary policy. Additionally, Stochastic can simply be taken to mean random, as in unpredictable. Lastly, volatility is another key concept and it is referring to how much something is changing. In our case, volatility usually refers to how stable a stock or stock index is.

Mathematically modeling financial markets and their derivatives can most notably be traced back to the Black-Scholes Model. This model is the traditional, established way to calculate theoretical option prices. It’s development lead to an increase in options trading as it showed that each option has a unique price, regardless of the return of the underlying asset or the risk associated with it. The assumptions of the model are twofold:

- The volatility \( \sigma \) of the market is constant, and
- The stock returns follow a log normal distribution \( \ln S_t \sim N(\mu, \sigma) \)

The call price equation:

\[
C = S_t N(d_1) - Ke^{-rt}N(d_2)
\]

\[
d_1 = (\ln (S_t/K) + (r + (\sigma^2/2) T) / (\sigma\sqrt{T})
\]

\[
d_2 = d_1 - \sigma\sqrt{T}
\]
\[ S = \text{current stock price}; \ K = \text{strike price}; \ r = \text{risk free rate}; \ T = \text{time to maturity}; \ N = \text{a normal distribution} \]

Under these conditions we can obtain a fair options price, however, looking at Figure 1 we can easily see that the assumption of constant volatility is not very realistic: The Volatility Index (VIX) is not constant at all. This Index tracks the aggregate volatility of the entire market. Out of the model’s that I discuss in this report, the Black-Scholes model is the only one where the underlying dynamics assume a constant volatility. Although unrealistic, it is simple and tractable.

![Time Series of SPY (Solid Line) and VIX (Dotted Line)](image)

*SPY: S&P 500 – a stock market index that is closely watched; VIX: Volatility Index – displays the volatility of the entire market.

And so, we have a model that better captures real market conditions. This model is the Heston model (Heston 1993). As the name implies, this model was first introduced by Steven Heston in 1993 and was a revolutionary step towards more realistic modeling. The primary claim put forward by Heston was that he treated the volatility of stock processes as random. The assumptions for this model are:

- The volatility of the market is stochastic (follows a random process), and
- The stock price process itself is also stochastic
The equation for the Heston Model, as well as the other models, can be seen in Figure 2.

This model is an improvement over the Black-Scholes model as it allows the volatility to be stochastic, but it still must assume a constant mean level of market variance. In other words, the volatility changes, but in the model, we assume a long run average which is constant (θ). This becomes a problem since we know that stock market behavior can change abruptly because of some economic or political shock to the market and the Heston model does not account for that. During these times the market can switch between a stable, low volatility state to an unstable, high volatility state. This is what we call regime switching.

A journal article by Mary Hardy in 2001 showed how to incorporate regime switching into the traditional Black-Scholes Model. She did this by having two sets of parameters. That is:

- \( \ln S_t \sim N (\mu_1, \sigma_1) \)
- \( \ln S_t \sim N (\mu_2, \sigma_2) \)

When the model is in one regime, it will operate under one set of parameters. When it switches to another regime, those set of parameters will change. In this way we try to simulate changing conditions in the markets.

Consequently, it is thought that one can incorporate this behavior into the Heston through the same process of having two sets of parameters, one for each regime.

E.g. Regime 1: \((\mu_1, k_1, \theta_1, \sigma_1)\)
Regime 2: \((\mu_2, k_2, \theta_2, \sigma_2)\)

Looking at the time series for the S&P 500 and VIX we can observe that the stock price and the corresponding volatility index are negatively correlated. Additionally, we can see that in the years between 1993 and 1997, the market stabilized in a low-level market volatility. From 1998 to 2003, the market volatility jumped up and stabilized in a high-level market volatility regime. In the years of 2004 to 2007, the market then went back and stabilized in the low-level volatility regime. Using this pattern of volatility changes characterizing different period of the
stock market, we can use the mean level of market volatility to define the regimes in our model. We also see from this example that regimes switch from time to time.

It is then natural to consider pricing models that allows the transition of regimes over time. Research shows that a Markov switching stochastic volatility framework better fits the actual data of the VIX. A Markov chain is a sequence of events where the probability of each new event occurring relies only on the previous state it had. This knowledge led to Elliot et al to providing a solution to incorporating regime switching into the classic Heston model in their 2016 paper:

In addition to the Heston model, we characterize different regimes by different levels of mean variance. Now, the mean level of variance is modified as a function of time. If there are n number of regimes, then our mean variance \((\theta_t)\) can take values from \(\theta_1, \theta_2, \ldots, \theta_n\). For my research project, this is the type of model we will be using.

III. Problem Statement & Goals

The challenge of this research project is incorporating the aspects of regime switching and stochastic volatility into an options pricing model. Therefore, our foremost goal is to implement a regime switching Heston-type stochastic volatility model in MATLAB and successfully generate an option price from it. Then we can create option price surfaces to analyze the model’s implications. We also hope to calibrate the model parameters to real market data. This would entail minimizing the difference between our model’s price and the actual market price to obtain the most probable actual parameters of the market, instead of
just assigning values to parameters. The first step to achieving these goals is assembling the model.

**IV. Methodology**

In order to derive the call option pricing formula, we followed Elliot et al (Elliot et al 2016).

For our regime-Switching Heston stochastic volatility pricing model, we use the following notation:

\[
S_t \quad \text{price of stock},
\]
\[
x_t \quad \text{logarithm of } S_t, \text{ i.e. } x_t = \ln S_t,
\]
\[
\nu_t \quad \text{volatility of } S_t,
\]
\[
r \quad \text{risk-free rate},
\]
\[
K \quad \text{strike price},
\]
\[
\tau \quad \text{maturity of option}
\]
\[
z \quad \text{unit vector whose } i\text{-th component is one, e.g., } z = [1 \ 0]: \text{the initial state in regime 1},
\]
\[
\Gamma \quad \text{transition-rate matrix of the regime, i.e. } \Gamma := (\gamma_{ij})_{i,j=1}^n
\]
\[
\gamma_{ij}\Delta t \approx \mathbb{P}\{\theta_{i+1} = j | \theta_t = i\}
\]

Note that the unit vector $z$ works as an indicator to tell which regime the market currently lies in. For example, $z = [1 \ 0]$ means that we have two regimes and regime 1 is where the current state lies in. For our simulations we started in regime 1. We also define the Markov transition matrix $\Gamma$, in which $\Gamma_{ij}$ means the transition rate or transition probability for a market state switching from $i$ to $j$ in an infinitesimal interval.

The pricing formula is as shown and returns the call option price. It is well-known that the solutions for the first-order time-dependent ODEs are easy to evaluate. In our equation, the two terms inside the parenthesis give us probabilities.
In our model we consider two-state regimes with transition matrix $\Gamma$

$$c(x, v, z, K, \tau) = e^x \left( \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[ i\phi (x - \ln K + r\tau) + \tilde{\beta}_0 v < \tilde{\Psi}(0, \tau)z, 1 \right] d\phi \right)$$

$$- \quad K e^{-r\tau} \left( \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[ i\phi (x - \ln K + r\tau) + \beta_0 v < \Psi(0, \tau)z, 1 \right] d\phi \right)$$

where

(1) $\beta(\cdot)$ and $\tilde{\beta}(\cdot)$ are deterministic functions.

(2) $\Psi$ and $\tilde{\Psi}$ are the solutions of first-order linear matrix ODEs with time-dependent coefficients.

In our model we consider two-state regimes with transition matrix $\Gamma$

$$\Gamma = \begin{bmatrix}
\text{Regime 1} & \text{Regime 2} \\
\text{Regime 1} & 1 - \gamma_{12} & \gamma_{12} \\
\text{Regime 2} & \gamma_{21} & 1 - \gamma_{21}
\end{bmatrix}$$

$\gamma_{12}$ is the probability of moving from regime one to regime two and $\gamma_{21}$ is the probability of moving from regime two to regime 1. The first regime is a low mean variance regime and regime two is a high mean variance regime. We used the following parameters to characterize the market.

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>$r$</th>
<th>$\rho$</th>
<th>$\sigma_v$</th>
<th>$\kappa$</th>
<th>$\theta^1$</th>
<th>$\theta^2$</th>
<th>$\gamma_{12}$</th>
<th>$\gamma_{21}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.05</td>
<td>-0.5</td>
<td>0.2</td>
<td>1.5</td>
<td>0.025</td>
<td>0.075</td>
<td>0.5</td>
<td>1</td>
</tr>
</tbody>
</table>

Note that $\theta^1$ is less than $\theta^2$, both of which are the constant mean variance in the Heston framework. We then computed the prices with the analytical formula mentioned above in MATLAB. The code to do so can be found in the Appendix.

**V. Results**

After we had the model developed, we ran the MATLAB code to generate the call option prices. We then obtained the price surfaces of the call options under different scenarios. On the
x-axis we placed the strike price $K$, the $y$-axis is maturity $\tau$, and the $z$-axis has the option price.

We chose to see how option prices vary with strike price and maturity as those are the variables one can control when picking an options contract. A decreasing price is displayed using an increasingly blue color and increasing price an increasing yellow color. Furthermore, from these relationships we can tell if our model is functioning properly by observing whether it follows standard logic when it comes to variables that affect option prices.

Figure 3 is the scenario where volatility is at 0.02, which would place it in our low-volatility regime. A low volatility state is generally more profitable as price movements and their directions can more easily be understood. In our model, this state produces the lowest option prices, as expected. It is easy to observe that the option prices become more costly when the strike price decreases or the time to maturity increases. This is also to be expected, which is a good first step in validating the model’s accuracy.
Figure 4 depicts an intermediate initial volatility of 0.05 which is between the low and high regime. In this state the market is riskier and therefore should have higher prices overall. This option price surface has the same relationships as the first scenario; however, the whole surface is shifted up slightly. Increasing the initial volatility has made the options more expensive at every price.
Figure 5 has an initial volatility of 0.1, a very high and consequently very risky state. In this scenario where the volatility is very high, we can see that, again, the price surface retains the same relationships between strike price, maturity, and option price and is shifted upwards because of an increased initial market volatility. This surface, along with Figure 4, implies that a high regime is risky and that the cost of protecting against unfavorable market conditions, such as high volatility, should be high. As a result, the option prices become more expensive.

VI. Future Work

In conclusion, we were successful in achieving one of our goals: Implementing option price surfaces for a regime-switching Heston model. The proof of these results can be found in figures 3, 4 and 5. These surfaces showed us that our model follows traditional thought while implementing more accurate model assumptions, albeit at the cost of increased complexity. Whether or not this increased complexity is deemed fit is a subject for future developments. Future work of this project could include finding implied volatility surfaces. These surfaces would be like those of the option price surfaces. Additionally, we would like to provide analyses on model performance using information criterion tests, calibrate the model with real market data to acquire accurate parameters, and to develop more general regimes beyond the control
of mean variance $\theta_t$. More general regimes would extend the regime switching to other parameters. Eventually we hope to be able to use our model to hedge regime-switching options.

Lastly, I would like to thank Dr. Junsen Tang for being my advisor and the CAM Program at the University of St Thomas for providing funding.
VII. Bibliography


The MATLAB code that generates the call option price is listed below. Code written by Dr. Tang. This can be copy and pasted into MATLAB, supplied parameters, and then it will function.

% PSI Function –

function f=PSI(tau,u,kappa,sigmav,rho,gamma12,gamma21,thetabar1,thetabar2,phi)
  % f=PSI(1,0.2,0.3,0.4,-0.5,1,0.5,0.025,0.075,1i)
  GAMMAprime=[1-gamma12,gamma12;gamma21,1-gamma21];
  THETAbare=[thetabar1,0;0,thetabar2];
  f=eye(2)*exp((GAMMAprime+kappa*betat(tau,u,kappa,sigmav,rho,phi)*THETAbare)*(tau-u));

% Beta-t Function –

function f=betat(tau,t,kappa,sigmav,rho,phi)
  % f=betat(1,0,3,0.4,-0.5,0.2);
  eta=sqrt((kappa-rho*sigmav.*phi).^2+sigmav^2.*phi.*phi+1i);
  gamma=(kappa-rho*sigmav.*phi+eta)./(kappa-rho*sigmav.*phi-eta);
  f=(kappa-rho*sigmav.*phi+eta)./sigmav^2.*(1-exp(eta.*(tau-t)))./(1-gamma*exp)

% Beta-t-tilde Function –

function f=betattilda(tau,t,kappa,sigmav,rho,phi)
  % f=betattilda(1,0,3,0.4,-0.5,0.2);
  etatilda=sqrt((kappa-rho*sigmav-1i.*rho*sigmav.*phi.^2+sigmav^2.*phi.*phi-li));
  gammatilda=(kappa-rho*sigmav-1i.*rho*sigmav.*phi+etatilda)./(kappa-rho*sigmav-1i.*rho*sigmav.*phi-etatilda);
  f=(kappa-rho*sigmav-1i.*rho*sigmav.*phi+etatilda)./sigmav^2.*(1-exp(etatilda.*(tau-t)))./(1-gammatilda.*exp(etatilda.*(tau-t)));

% Call Price Function –

function call=HestonCall(S0,v,K,r,tau,kappa,sigmav,rho,gamma12,gamma21,thetabar1 ,thetabar2,z)
  for n=1:1:20000
    dphi=0.005;
    phi(n)=eps;
s1(1)=0;
s2(2)=0;

s1(n+1)=s1(n)+exp(li.*phi(n)*(log(S0)-log(K)+r*tau)+betatildetau(n,0,kappa,sigmav,rho,phi(n)))*v.*(PSItildetau(n,0,kappa,sigmav,rho,gamma12,gamma21,thetabar1,thetabar2,phi(n))*z')*[1;1]./(1i.*phi(n))*dphi;
s2(n+1)=s2(n)+exp(li.*phi(n)*(log(S0)-log(K)+r*tau)+betat(tau,0,kappa,sigmav,rho,gamma12,gamma21,thetabar1,thetabar2,phi(n)))*v.*(PSI(tau,0,kappa,sigmav,rho,gamma12,gamma21,thetabar1,thetabar2,phi(n))*z')*[1;1]./(1i.*phi(n))*dphi;

end

P1=0.5+(1/pi)*real(s1(20001));
P2=0.5+(1/pi)*real(s2(20001));
call=S0*P1-K*exp(-r*tau)*P2;