CAM Report

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31 August 2015

Abstract

Buoyant force behaves similarly to spring force in that both can be represented as functions dependent on position. An object that undergoes these types of forces is said to exhibit a force profile. The intent of this project was to start with a known force profile and create an object that would reflect that specific profile.

1 Concept

The idea behind this research is that, given a specific force profile, we can design and build an object that will exhibit a buoyant force exactly the same as that profile. An example force profile can be seen below:

\[ F = 4x^3 + 9x^2 - 2x \]  \hspace{1cm} (1)

where \( x \) is the displacement of the object and \( \vec{F} \) is the corresponding force. We know that buoyant force is similar to spring force, as can be seen in equations 2 and 3.

\[ F = -\rho g V \]  \hspace{1cm} (2)

the equation for buoyant force

\[ F = -kx \]  \hspace{1cm} (3)

the equation for spring force
Although not inherently obvious, this relationship may better be seen if volume is written as the integral of area as a function of height.

\[
\vec{F} = -\rho g \int_{h_1}^{h_2} A(z) \, dz
\]  

(4)

Figure 1: Variable Diagram

2 Important Insights

Before beginning the experiment, there were a few details we had to keep in mind. Firstly, the force profile, \( F \), is a function of \( x \), which is the actual physical displacement of the object into the water. This is not always the same as the total depth that the object is submerged, represented by \( z \). When the container is small in size relative to the object being submerged, the rise in water level becomes non-negligible. This is relevant because other functions which are derived later are functions of the total depth. Refer to figure 1 for a visual representation of the variables we used.

3 Potential Problems

Throughout our research, we also encountered and dealt with several roadblocks to proper data collection and analysis. One such problem is surface
tension. Because our data collection is on a relatively small scale (i.e. a thousandth of a Newton), surface tension largely affects the force readings. Another issue is the porousness of the object being submerged. Using an object that is likely to absorb a significant amount of water during an experiment will drastically skew the data.

Figure 2: The Apparatus: A - Beaker, B - Balance, C - Jack, D - Suspended Object, E - Force Sensor, F - Height Gauge

4 Initial Steps and Setup

Before we could get started, we had to make sure we had the right tools for the trade. We needed something to measure the displacement of the object and a way to measure the change in force. We chose at first to use a
mechanical height gauge, a jack, and both a scale and Vernier force sensor to be certain that our force measurements were accurate. Later, our method changed to using a dial indicator instead of the height gauge, plus a wooden plank to maintain contact with the indicator. This is shown in figure 2, with a metal plate in place of the wooden plank. We first performed an initial test run using an aluminum cylinder in a large beaker of water to make sure we had the proper instruments. Next, it was time to apply the force profile principle to more complex shapes - specifically non-prisms.

5 Math

The simplest case to examine was that of prismatic objects in large containers. In this case, the rise in water level would be negligible and the cross-sectional area of the suspended object would be constant throughout the whole experiment. However, there are three other cases to consider: a non-prism in a large container, a prism in a small container, and a non-prism in a small container. The challenges arose when we had to work with a rising water level and/or a changing cross-sectional area. Through both integral calculus and algebra, we arrived at the conclusion that there is no closed-form solution to solve this issue of the varying cases. However, we were able to use the conservation of water in the container, along with the derivative of buoyant force, to derive equations necessary for finding the change in water level and the cross-sectional areas. Because the volume of water never changes, only the placement in the container as the object is submerged, we are able to write this formula (see equation 5):

\[ V_d = V_s \] (5)

*The volume of the water displaced is equal to the volume of the object submerged.*

This equation can be broken down using the relationship of volume as being the integral of area over a change in height:

\[ \int_{x_1}^{x_2} A(x)dx = \int_{w_1}^{w_2} (A_C - A(w))dw \] (6)

where \( A(x) \) is the cross-sectional area of the object as a function of displacement \( x \), \( A_C \) is the cross-sectional area of the container, and \( A(w) \) is the cross-sectional area of the water as a function of the rising water level.
For prismatic objects, the cross-sectional area will remain constant. Therefore:

\[ xA = w(A_C - A) \]  \hspace{1cm} (7)

Even for non-prisms, when observed on smaller scales over a small change in height, the cross-sectional area will be constant. Then, the equation can be simplified:

\[ dxA = dw(A_C - A) \]  \hspace{1cm} (8)

Also, keep in mind that the total depth submerged is equal to the sum of the displacement and the rise in water level.

\[ z = x + w \]  \hspace{1cm} (9)
\[ dz = dx + dw \]  \hspace{1cm} (10)

Thus,

\[ dxA = dz(A_C - A) - dx(A_C - A) \]  \hspace{1cm} (11)
\[ dxA_C = dz(A_C - A) \]  \hspace{1cm} (12)
\[ \frac{dxA_C}{A_C - A} = dz \]  \hspace{1cm} (13)

Starting with equation 12, we can derive an equation to find the cross-sectional area of the object for a change in displacement.

\[ \frac{dx}{dz} = 1 - \frac{A}{A_C} \]
\[ \frac{dF}{dz} = \rho g A \]  \hspace{1cm} (14)
\[ (15) \]
\[
\frac{dx}{\rho g} = 1 - \frac{A}{A_C} \quad (16)
\]
\[
dx = \frac{dF}{\rho g A} - \frac{dF}{\rho g A_C} \quad (17)
\]
\[
dx = \frac{dF}{\rho g} \left( \frac{1}{A} - \frac{1}{A_C} \right) \quad (18)
\]
\[
\rho g dx \frac{dF}{dF} = 1 - \frac{1}{A} \quad (19)
\]
\[
A \rho g dx \frac{dF}{dF} = 1 - \frac{A}{A_C} \quad (20)
\]
\[
A \rho g dx 1 \frac{dF}{dF} + \frac{A}{A_C} = 1 \quad (21)
\]
\[
A \left( \rho g dx \frac{dF}{dF} + \frac{1}{A_C} \right) = 1 \quad (22)
\]
\[
A = \frac{1}{\rho g + \frac{dF}{dx} \frac{1}{A_C}} \quad (23)
\]
\[
A = \frac{\frac{dF}{dx}}{\rho g} \quad (24)
\]

However, for an object being submerged in a container much larger in size, the equation simplifies to the following:

\[
A = \frac{\frac{dF}{dx}}{\rho g} \quad (25)
\]

### 6 Early Experiments

After discovering these equations, we then verified them by predicting the cross-sectional areas of objects which had easily-discernible cross-sectional areas outside of the buoyant force experiment. If our calculated cross-sectional areas aligned with the derived cross-sectional areas of our tested object, our math would be correct. We initially tested this using a cylinder being suspended into the water on one of its circular faces. That way, the cross-sectional area would be constant throughout our testing. In order to change \( x \) in a controlled fashion, we used a height gauge to find the next point from which we were to take measurements. We would then move a jack to the new
height. The jack would then move a scale with a container of water closer to our tested object, further submerging the stationary object. Our tested object was attached to a force sensor, which would read the force on it as the object was further submerged into the water. A second way of measuring the change in force was by using the scale as it read the change in mass of the water due to the object being submerged. Our first experiment worked both to see if our theoretical math worked out experimentally, and also that our setup functioned well. The data for this experiment can be seen in figure 3. From here, we were ready to try more complicated objects.

![Figure 3: The Force Profile for the Cylinder](image)

7 Intermediate Experiments

To further test our equations, we decided to test an object that has changing cross-sectional areas, or a non-prism. This would further validate our equations if they derived the correct cross-sectional areas at varying heights. It was at this point we started to run into problems. To test our math on an object that changes cross-sectional area with height, we decided to use
Figure 4: Wooden Block Experiment
a wooden disk, but to submerge it on its round side so that it would have varying rectangular areas. Our data looked rocky from surface tension but soon started to follow our theoretical data for a few points. Not long after, the data began to plateau instead of increase and decrease as it would be expected for a disk. Upon further investigation, we realized what was going on with our experiment. First, our initial force reading started high, then quickly fell and grew again. This was due to the surface tension of the water. We could actually see the water rising up the object at the objects entry, causing the force readings to be higher from the pull of the water. Another issue was the wood getting soaked from submersion. As the wood was lowered more into the beaker, more of the object would get wet and soak up water into the rest of the disk, skewing the data by adding extra weight (see figure 4). In our following experiment, we solved this problem by using a metal disk, which was not nearly as porous as the wooden disk. To combat the initial surface tension, which decreased considerably after the object was submerged, we started data collection after the object was already partially submerged in the water. The majority of the surface tension issues occurred when initially submerging the object into the water, so this was a way to largely avoid it. We also tried dissolving dish soap into the water as a surfactant, but found this to be less effective. However, it is worth trying again in future experiments, remembering that it will change the density, \( \rho \), used in calculations.

8 Uncertainty Analysis

To make sure that our data was aligning well with our predictions, we performed uncertainty analysis with the use of partial derivatives. This was done by taking the partial derivatives of our variables with respect to the cross-sectional area. This would illustrate how having mistakes in measurements of one variable would affect the area. Upon taking the magnitude of all these partial derivatives, we were able to discern maximum and minimum values that our experimental data could be within and still be considered reasonable. From there we deduced that if our theoretical data is within the uncertainty margins, it is safe to assume that our data is reasonably correct.
9 3D-Printing

The ultimate step to validating our mathematical theory was to actually derive a force profile, design an object using the cross-sectional area equation, print and test the object, and compare its theoretical force profile with the empirical data.

9.1 Force Profile

To derive a force profile, we started with an arbitrary graph within known parameters, following that the profile would be constantly increasing and positive as well as having a 0-Newton force when the displacement was 0. We decided that a quartic function within these parameters might make for an interesting object. Taking derivatives, solving for constants, integrating the functions, and experimenting with different constant values left us the following force profile:

\[ F = \frac{1}{5} \left( \frac{x^4}{24} - \frac{x^3}{3} + \frac{3x^2}{4} \right) + \frac{1}{50}x \]  

(26)

9.2 Designing the Objects

Knowing that container size made a significant impact on the force profile of an object, we decided then to make two objects that would exhibit the same force profile; however, one such object would reflect the profile while submerged in a smaller container while the other would be submerged in a large container. Since we knew the forces and at what displacement they occurred, we could easily ascertain the cross-sectional areas we needed using equations 13 and 24. Equation 13 is particularly useful in the case of the rising water level, since the cross-sectional area is dependent on \( z \), not on \( x \).

9.3 Printing

Next, we decided our areas should represent circles, although they could be any shape. Using the found areas, we solved for the radii and entered the information into an Excel document properly formatted to be imported as a data table in SolidWorks. We then created the arbitrary shape of lofted circles and applied the data tables in order to create our objects, which we
named Large and Small after their container sizes. Next, we sent the STL files to be printed.

![The Two Objects: Small (left) and Large](image)

**Figure 5: The Two Objects: Small (left) and Large**

### 9.4 Testing and Comparing

After our two objects were printed, we then tested them in their appropriate containers. We found that the smaller object output data that aligned very well with the expected force profile, as shown in figure 6.

Unfortunately, the large object did not have the force profile expected (see figure 7).

However, we later realized that we’d entered the wrong loft distances into SolidWorks, having accidentally used the same formula for the small part as the large. The area equation for the large part, because the rise in water level is negligible, is only dependent on $x$. We then re-derived what the force...
profile would be for the object we’d printed, and found the profiles to match fairly well, shown in figure 8 at the end of the document.

10 Future Steps

There are a few directions we propose for the future of this project. Firstly, we are writing an article for *The Physics Teacher* journal. Hopefully our article will be accepted and can be a useful resource for physics teachers to create labs based on our findings or following our experimental layout. Additionally, it might be very interesting to experiment with changing the shape of the container, or to use a container with a varying cross-sectional area. With these findings, the next step might be to design a non-linear spring using buoyant force. This could be implemented using water and air, two or more liquids of varying density, or more generally materials with a noticeable change in pressure as a function of depth. A prism in an irregular container or a non-prism in a regular container could be suspended by a rod, and this assembly would become a spring. It is important to keep in mind,
however, that this apparatus would have to operate vertically as gravity is a main factor of buoyant force. In cases where potentially long-lasting, non-linear springs are necessary, this type of buoyant-force spring could be very useful.
Figure 8: The Real Force Profile for Large 3-D Printed Object