Chaotic Flow in the Lorenz Model

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What is Chaos???

- In mathematical terms, a chaotic system is sensitive to initial conditions.
- Behavior will appear "random" to the uneducated observer.

Chaos Theory

- Chaos Theory grew out of topics in nonlinear dynamics such as the three-body problem and fluid flow.
- Although has been a topic of interest since the early 1900's, chaos theory was not able to be studied in great detail until the computer was invented.
- As a relatively new branch of mathematics, chaos theory is an active area of research today.

The Navier-Stokes Equations

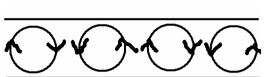
$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{\nabla P}{\rho} + \nu \nabla^2 \mathbf{u} + \frac{\mathcal{F}}{\rho},$$

$$\frac{\partial \rho}{\partial t} = -\rho \nabla \cdot \mathbf{u}.$$

- The Navier-Stokes equations describe fluid flow of an incompressible fluid.
- They are derived from conservation principles of energy, mass, momentum and angular momentum.
- The Lorenz Model is obtained by making many simplifications and generalizations from the Navier-Stokes equations.

The Rayleigh-Benard Cell

- A container filled with fluid that heats one end and cools the other. The behavior exhibited becomes chaotic when the temperature difference is above a critical value.
- It can be thought of as similar to the atmosphere heated by the sun and cooled by the sea.



The Lorenz Model

$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = x(\tau - z) - y$$

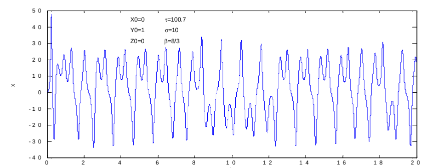
$$\frac{dz}{dt} = xy - \beta z$$

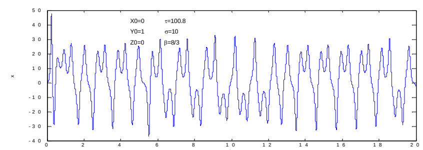
•Derived from greatly simplifying the Navier-Stokes equations, specifically in the case of a Rayleigh Benard Cell.

•In the simplification, the physical meaning of the system is lost, however, the long term behavior exhibited by the system shows similar patterns to those of the atmosphere.

History

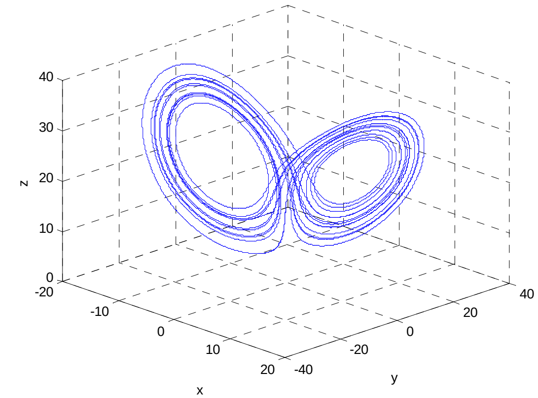
•In 1963, MIT meteorologist Edward Lorenz came up with these equations to model the atmosphere. One day, he was surprised by results that he came up with, so he decided to re enter the data in the computer half way through the run, rounding the initial conditions from six decimal places to three. He found that this tiny difference in initial conditions caused a huge difference in long term behavior, as is shown below.





"The Butterfly Effect"

- Lorenz coined the phrase "butterfly effect" describing how a slight change in initial conditions can lead to drastically different outcomes. "The flap of a butterfly's wings in South America could be responsible for a tornado in Texas."
- He chose a butterfly because a three dimensional image of the trajectory mapped out by his equations looks like a butterfly.



What does this mean for future weather predictions?

- Because so much simplification is made in the derivation of these equations, actual physical interpretation of this system is lost. However, the underlying chaos of the systems is so rich, that understanding it will prove most valuable in future weather models.

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