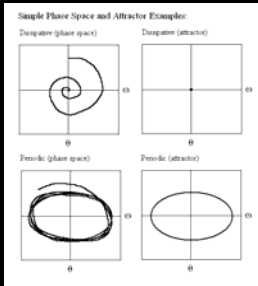


Non-Linear Dynamics: Lyapunov Exponents

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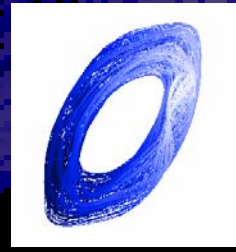
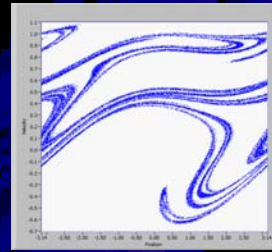
Attractors and Attractor Reconstruction:

Attractors:



Trajectories in phase space converge toward attractors. Simple, 'normal' examples of attractors include points, limit cycles, and saddle points. Chaotic Attractors are where it gets interesting. They are called 'strange attractors' and have a fractal structure and non-integer dimension

Are these chaotic attractors?



Poincaré section movies display 'stretching' and 'folding' behavior necessary for chaotic processes...

But time-delay reconstructed attractors are conducive for analysis. (Poincaré and reconstructed attractor from same data)

Lyapunov Exponents:

Quantifying Chaos

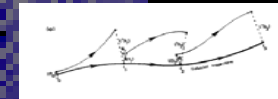
Lyapunov Exponent Defined:

$$\epsilon(n) \approx \epsilon e^{n\lambda}$$

Lyapunov Exponents, often denoted by λ , give the average rate of divergence between neighboring trajectories in an attractor, the 'stretching' character of the attractor. This provides a quantitative measure of the extreme sensitivity to initial conditions, indicative of chaos.

There is a spectrum of exponents corresponding to the number of independent variables in the system, but we are most concerned with the dominant, 1st exponent. A Lyapunov exponent can be +, -, 0 depending on the behavior. For example, a dissipative run would have a negative dominant Lyapunov exponent (trajectories converge). Periodic behavior would have a zero dominant Lyapunov exponent, (trajectories neither converge or diverge). Chaotic behavior has a positive dominant Lyapunov exponent.

Wolf's Algorithm:



$$\lambda_1 = \frac{1}{t_M - t_0} \sum_{k=1}^M \log_2 \frac{L'(t_k)}{L(t_{k-1})}$$

Wolf's programs average individual locally calculated Lyapunov exponents from time $t=0$ to M on a time-delay reconstructed attractor. Time-delay attractor reconstruction proved to be an integral part of this project.

The Lyapunov Exponent calculation has to be built as an average of local divergences because of the 'folding' nature of the attractor. If trajectories are followed too far forward in time, they may come back close together again, skewing the measured divergence and resulting Lyapunov exponent calculation.

Why Attractor Reconstruction:

Our position data is unbounded (i.e. the pendulum is free to spin in one direction) so analyzing nearby trajectories in regular phase space would be unfeasible. We could analyze folded position data, similar to how we make Poincaré plots, but it has discontinuities

The velocity time series, however, doesn't have these problems and is readily derived from experimental data with the Java analysis package. As a result, the velocity series is a good candidate for analysis. Wolf's programs 'BASGEN' and 'FET' are written to utilize such a time series, and with slight modification can work on our data.

Time Delay Attractor Reconstruction

Thankfully, although a time-delay reconstructed attractor is different than the actual underlying attractor, it has the same properties such as Lyapunov exponents and dimension measurements.

What we want to do is build an attractor from a time series of one variable, angular velocity in our case. There are two significant parameters that dictate the process, time delay (τ) and embedding dimension.

For example, given a time series of one variable:

$$(x_0, x_1, x_2, \dots, x_n)$$

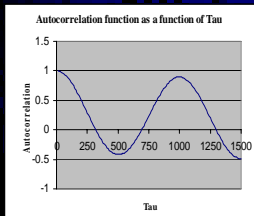
A time delay reconstruction for a given delay time (τ), and embedding dimension (d), would be:

$$\left((x_0, x_{0+\tau}, x_{0+2\tau}, \dots, x_{(d-1)\tau}), (x_1, x_{1+\tau}, x_{1+2\tau}, \dots, x_{1+(d-1)\tau}), \dots \right)$$

Time Delay (τ)

A correlation function can help in choosing a reasonable value for Time Delay (τ).

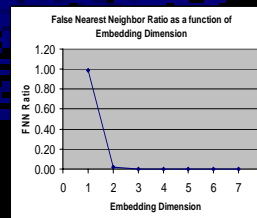
$$\frac{\sum_{i=1}^{N-m} (x_i - \bar{x})(x_{i+\tau} - \bar{x})}{\sum_{i=1}^N (x_i - \bar{x})^2}$$



Embedding Dimension:

The idea of False Nearest Neighbors can help determine adequate embedding dimension.

$$\frac{|x(n+\tau) - x'(n+\tau)|}{R_d(n,\tau)} > R_{tol}$$



Conclusions:

Through this investigation I have learned much about chaotic attractors, attractor reconstruction, and Lyapunov exponents, although there is much more material worthy of investigation. I have acquired a better understanding of appropriate parameter values, namely τ and the embedding dimension, for time-delay attractor reconstruction, but the resulting Lyapunov exponent calculation is not as parameter independent as I would have hoped.

I would estimate the dominant Lyapunov exponent for our system to be ~ 2 bits/sec. We lose the ability to predict what our system will do at a rate of ~ 2 bits a second.

Sources:

- The following materials were integral in preparing this poster:
- Barker, G.L, and Gollub, J.P. *Chaotic Dynamics an introduction* Cambridge University Press, 1996.
- R. Regger, H. Kantz, and T. Schreiber, *Practical implementation of nonlinear time series methods: The TISEAN package*, CHAOS 9, 413 (1999)
Available online: http://www.mpipks-dresden.mpg.de/~tisean/TISEAN_2.1/docs/indexf.html
- Kennel, Matthew B., Brown, Reggie, Abarbanel, Henry D.I., "Determining embedding dimension for phase-space reconstruction using a geometrical construction" *Physical Review A*, Vol 45, Num 6, 15 March 1992, 3403-3411.
- Williams, Garnett P., *Chaos Theory Tamed*, Joseph Henry Press, Washing D.C. 1997.
- Wolf, Alan, Swift, Jack B., Swinny, Harry L. Vastano, John A. "Determining Lyapunov Exponents from a Time Series," *Physica 16D*, (1985), 285-317.