Zero Divisor Graphs of Commutative Rings

Michael Driscoll\textsuperscript{a}  Mitchell Klein\textsuperscript{b}  Alexandra Ubel\textsuperscript{c}

\textsuperscript{a}University of St. Thomas, 2115 Summit Avenue St. Paul, MN  
\textsuperscript{b}University of St. Thomas, 2115 Summit Avenue St. Paul, MN  
\textsuperscript{c}University of St. Thomas, 2115 Summit Avenue St. Paul, MN
Abstract. In this paper we will investigate the interactions between the zero divisor graph, the annihilator class graph, and the associate class graph of commutative rings.

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1 Introduction

We begin by providing the definitions and notation used throughout the paper. In the second section we discuss the zero divisor graphs of certain finite commutative rings. The third section covers zero divisor graphs of commutative rings constructed by the direct product of infinite fields and commutative rings.

Given a commutative ring $R$, an element $z \in R$ is a zero divisor if there exists a nonzero $y \in R$ such that $xy = 0$. We denote the set of zero divisors as $Z(R)$, and the set of nonzero zero divisors denoted by $Z^*(R)$. For $x \in R$, the annihilator of $x$, denoted $\text{ann}(x)$, is $\{y \in R | xy = 0\}$ and we say $x \sim \text{ann} y$ if and only if $\text{ann}(x) = \text{ann}(y)$. It can be shown that the annihilator of any element in a ring is an ideal. We define the annihilator class of $x$, denoted $[x]_{\text{ann}}$, as $[x]_{\text{ann}} = \{k \in R : \text{ann}(k) = \text{ann}(x)\}$ and the set of all annihilator classes denoted by $\overline{Z}(R)$. An element $x$ is nilpotent if $x^n = 0$ for some $n \in \mathbb{N}$. The set of all units in $R$ is denoted by $U(R)$. If $x, y \in R$ where $R$ is a commutative ring, we say $x$ and $y$ are associates, denoted $x \equiv y$, if and only if $(x) = (y)$. The associate class of $x$, denoted $[x]_{\text{assc}}$, is $[x]_{\text{assc}} = \{k \in R | (k) = (x)\}$, and the set of all associate classes is denoted by $\overline{Z'}(R)$. A ring $R$ is a local ring if and only if $R$ has a unique maximal ideal. For a graph $G$, we define $V(G)$ and $E(G)$ to be the sets of vertices and edges of $G$, respectively. Two elements $x, y \in V(G)$ are defined to be adjacent, denoted by $x \sim y$, if there exists an edge between them. A path between two elements $v_1, v_n \in V(G)$ is an ordered sequence of distinct vertices of $G$, $\{a_1, a_2, \ldots, a_n\}$, such that $a_{i-1} \sim a_i$ for $1 \leq i \leq n$. The length of a path between $x$ and $y$ is the number of edges crossed to get from $x$ to $y$ in the path. The distance between $x, y \in G$, denoted $d(x, y)$, is the length of a shortest path between $x$ and $y$, if such a path exists; otherwise, $d(x, y) = \infty$. For the purposes of this paper, we define $d(x, x) = 0$. The diameter of a graph is $\text{diam}(G) = \max\{d(x, y) | x, y \in V(G)\}$. An element $v \in V(G)$ is said to be looped if there exists and edge from $v$ to itself. A graph $G$ is called complete bipartite if there exist partitions $A, B \subset V(G)$ such that $A \cup B = V(G)$, $A \cap B = \emptyset$, for all $a_i, a_j \in A$ and $b_i, b_j \in B$, $i \neq j$, we have $a_i \not\sim a_j$ and $b_i \not\sim b_j$, and for all $a_i \in A$ and $b_j \in B$, we have $a_i - b_j$. Finite complete bipartite graphs are denoted as $K_{m,n}$, where $|A| = m$ and $|B| = n$. A graph $G$ is said to be complete bipartite reducible if and only if there exists a complete bipartite graph $G'$ such that $V(G') = V(G)$ and $E(G') \subseteq E(G)$. If two graphs are isomorphic, they must have the same number of vertices, the same number of edges, the same degrees for corresponding vertices, the same number of connected components, and the same number of loops. A graph $G$ is a star graph if $G = K_{1,n}$. A graph $G$ is said to be star-shaped reducible if and only if there exists a vertex $g \in V(G)$ such that $g$ is adjacent to all other vertices in $G$ and $g$ is looped. More information about graph theory may be found in [6].

2 New Definitions

We now consider graphs of the finite rings $\mathbb{Z}_p^2$ and $\mathbb{Z}_p[x]/(x^2)$, for $p$ a prime. We will show that these non-isomorphic rings have the same zero divisor, annihilator class, and associate class graphs. Initially, we sought to prove that if two sets of zero divisor, annihilator class
and associate class graphs were isomorphic, then the rings from which the graphs came would also be isomorphic. However, Theorem 2.3 illustrates why this is not possible. In order to move on to the results it is first necessary to prove that associates and annihilators are equivalence relations.

We define the zero divisor graph of $R$, denoted by $\Gamma(R)$, as follows: $V(\Gamma(R)) = Z^*(R)$, and $x \sim y$ if and only if $xy = 0$. We define the annihilator class graph of $R$, denoted $\overline{\Gamma}(R)$, as follows: $V(\overline{\Gamma}(R)) = \{0\}_{ann}$, and $[x]_{ann} \sim [y]_{ann}$ if and only if $xy = 0$. Finally, we define the associate class graph of $R$, denoted $\Gamma'(R)$, as follows: $V(\Gamma'(R)) = Z'(R) \setminus \{0\}$, and $[x]_{assc} \sim [y]_{assc}$ if and only if $xy = 0$. We now show that these edges and vertices are well-defined.

Let $R$ be a commutative ring with identity and let $ab = 0$, $a \sim_{ann} c$, and $b \sim_{ann} d$. Since $ab = 0$ then $b \in ann(a)$ since $a \sim_{ann} c$ then $b \in ann(c)$ therefore $bc = 0$. Since $bc = 0$ then $c \in ann(b)$ and $b \sim_{ann} d$ then $c \in ann(d)$ therefore $cd = 0$. Thus, the $\Gamma$ edge definition is well defined.

Let $R$ be a commutative ring with identity where $ab = 0$, $a \sim_{assc} c$, and $b \sim_{assc} d$. Since $a \sim_{assc} c$, there exists $x, y \in R$ such that $ax = c$ and $cy = a$. Now, since $b \sim_{assc} d$, there exists $x', y' \in R$ such that $bx' = d$ and $dy' = b$. So $cd = d(ax) = (ax)(bx') = (xx')(ab) = 0$, therefore, $cd = 0$. Thus, the $\Gamma'$ edge definition is well defined.

**Lemma 2.1.** The relation $\sim_{ann}$ is an equivalence relation.

**Proof.** Recall that the definition of an annihilator of $a$ is $ann(a) = \{x \in R | ax = 0\}$. Clearly $\sim_{ann}$ is reflexive. Symmetry also holds because for every $a, b \in R$, $a \sim_{ann} b$ implies $b \sim_{ann} a$ because if $\{x \in R | ax = 0\} = \{y \in R | by = 0\}$ then $\{y \in R | by = 0\} = \{x \in R | ax = 0\}$ by symmetry of set equality. Finally $\sim_{ann}$ is transitive because for every $a, b, c \in R$, if $a \sim_{ann} b$ and $b \sim_{ann} c$, then $\{x \in R | ax = 0\} = \{y \in R | by = 0\}$ and $\{x \in R | bx = 0\} = \{y \in R | cy = 0\}$. Thus by set equality $\{x \in R | ax = 0\} = \{y \in R | cy = 0\}$. $\square$

**Lemma 2.2.** The relation $\sim_{assc}$ is an equivalence relation.

**Proof.** Recall from the definition of associates that we say $x \sim y$ if and only if $(x) = (y)$. Clearly associativity is reflexive. Symmetry is also preserved because for every $a, b \in R$, if $a \sim b$ then $(a) = (b)$ and thus $(b) = (a)$. Hence $b \sim a$. Finally there is transitivity of associates. For every $a, b, c \in R$, if $a \sim b$ and $b \sim c$ then $(a) = (b)$ and $(b) = (c)$ therefore $(a) = (c)$. Because $\sim$ preserves reflexivity, symmetry, and transitivity $\sim$ is an equivalence relation. $\square$

The previous work allows elements of rings to be grouped into annihilator and associate classes. The graphs of these classes will greatly simplify the zero divisor graphs of any ring.

## 3 Graphs of Finite Rings

We now examine the graphs of finite commutative rings. Below are listed examples of the graphs from $\mathbb{Z}_{12}$ and $\mathbb{Z}_4[x]/(x^2 + 2x)$. Figure 1 displays the three graphs of $\mathbb{Z}_{12}$ that are discussed in this paper. As shown $\Gamma(\mathbb{Z}_{12}) \cong \Gamma'(\mathbb{Z}_{12})$. 

Theorem 3.1. Let $p$ be a prime. Then $\Gamma(R_1) \cong \Gamma(R_2)$, $\bar{\Gamma}(R_1) \cong \bar{\Gamma}(R_2)$, and $\Gamma'(R_1) \cong \Gamma'(R_2)$.

Proof. Consider $R_1$. Since $Z^*(R_1) = \{pk | 1 \leq k \leq p-1\}$, $|V(\Gamma(R_1))| = p-1$, and we have $\Gamma(R_1) = K^{p-1}$.

Note that $\text{ann}(p) = \{y \in R_1 | p \cdot y = 0\} = \{pk | 0 \leq k \leq p-1\}$, since $p(kp) = k^2p = 0$. Thus $\text{ann}(p) = Z(R_1)$. Let $0 \leq k \leq p-1$. Observe $\text{ann}(kp) = \{y \in R_1 | kp \cdot y = 0\} = \{mp | 0 \leq m \leq p-1\} = Z(R_1)$. Thus $p_\text{ann} kp$ since $\text{ann}(p) = \text{ann}(kp)$ and $V(\Gamma(R_1)) = \{\llbracket p \rrbracket_{\text{ann}}\}$ where the single vertex is looped to itself because $p^2 = 0$.

Since $Z^*(R_1) = \{pk | 1 \leq k \leq p-1\}$ and $k \in U(R_1)$, there exists $k^{-1} \in R_1$ with $(kp)(k^{-1}) = p$. Thus and $p_\text{assoc} kp$ for all $1 \leq k \leq p-1$. Thus $V(\Gamma'(R_1)) = \{\llbracket p \rrbracket\}$ where $[p] = Z^*(R_1)$ and $\Gamma'(R_1) = K^1$ where the single vertex is looped to itself because $(p^2) = 0$.

Next consider $R_2$. Since $Z^*(R_2) = \{kx | 1 \leq k \leq p-1\}$, $|V(\Gamma(R_2))| = p-1$, and we have $\Gamma(R_2) = K^{p-1}$.

Note that $\text{ann}(x) = \{y \in R_2 | x \cdot y = 0\} = \{xk | 0 \leq k \leq p-1\}$, since $x(kx) = kx^2 = 0$. Thus $\text{ann}(x) = Z(R_2)$. Let $0 \leq k \leq p-1$, observe $\text{ann}(kx) = \{y \in R_2 | kx \cdot y = 0\} = \{mx | 0 \leq m \leq p-1\} = Z(R_2)$. Thus $x_\text{ann} kx$ since $\text{ann}(x) = \text{ann}(kx)$ and $V(\Gamma(R_2)) = \{\llbracket x \rrbracket_{\text{ann}}\}$ where the single vertex is looped to itself because $x^2 = 0$.

Since $Z^*(R_2) = \{kx | 1 \leq k \leq p-1\}$ and $k \in U(R_2)$, there exists $k^{-1} \in R_2$ with $(kx)(k^{-1}) = x$. Thus $x_\text{assoc} kx$ for all $1 \leq k \leq p-1$. Thus $V(\Gamma'(R_2)) = \{\llbracket x \rrbracket\}$ where $[x] = Z^*(R_2)$ and $\Gamma'(R_2) = K^1$ where the single vertex is looped to itself because $x^2 = 0$. 

Figure 1: $\Gamma(\mathbb{Z}_{12}), \bar{\Gamma}(\mathbb{Z}_{12}), \Gamma'(\mathbb{Z}_{12})$

It is not always the case that $\bar{\Gamma}(R) \cong \Gamma'(R)$, as can be shown in Figure 2 with the graphs of $\mathbb{Z}_4[x]/(x^2 + 2x)$.

Figure 2: $\Gamma(\mathbb{Z}_4[x]/(x^2 + 2x)), \bar{\Gamma}(\mathbb{Z}_4[x]/(x^2 + 2x)), \Gamma'(\mathbb{Z}_4[x]/(x^2 + 2x))$
Thus, $\Gamma(R_1) \cong \Gamma(R_2), \tilde{\Gamma}(R_1) \cong \tilde{\Gamma}(R_2),$ and $\Gamma'(R_1) \cong \Gamma'(R_2)$. \hfill \Box$

Theorem 3.1 shows that the three graphs featured in this paper do not always distinguish non-isomorphic rings from each other.

4 Graphs of Infinite Rings

In this section we will be examining graphs of infinite rings, direct products of infinite fields and finite rings, and direct products of infinite fields and finite fields. We will show that some of these non-isomorphic structures have the same associate class graph. Let $F$ be an arbitrary infinite field and let $F_q$ be an arbitrary field of order $q^a$, where $q$ is prime.

The next result was borne in an attempt to distinguish the rings from the previous theorem by using the zero divisor graphs of commutative rings which are isomorphic to the direct product of a ring and a field. It shows that the associate graph of $\mathbb{F} \times \mathbb{F}_q[x]/(x^2)$ is congruent to the associate graph of $\mathbb{F} \times \mathbb{F}_q[x]/(x^2)$ for primes $p$ and $q$.

**Theorem 4.1.** Let $R_1$ and $R_2$ be commutative rings with $R_1 = F \times \mathbb{Z}_{p^2}$ and $R_2 = F \times \mathbb{F}_q[x]/(x^2)$ for an infinite field $F$ and primes $p$ and $q$. Then $\Gamma'(F \times \mathbb{Z}_{p^2}) \cong \Gamma'(F \times \mathbb{F}_q[x]/(x^2))$.

**Proof.** Consider $R_1$. Let $m, n \in U(\mathbb{Z}_{p^2})$, and observe that $m \sim n$ because $(m)$ and $(n)$ both generate $\mathbb{Z}_{p^2}$. Observe that $0 \sim (0,m)$. Thus $(0,1) \sim (0,m)$ and we have the $[(0,1)]$ associate class.

Observe that $\mathbb{Z}_{p^2}$ is a local ring and therefore has a maximal ideal which is $(p) = \{0, p, 2p, \ldots, (p-1)p\}$. Because $a$ is a unit, it will equal the ideal generated by any other unit in $R_1$. Thus $(ap) = (p)$ for $a \in U(\mathbb{Z}_{p^2})$ and thus $ap \sim p$ and in $R_1$. So, $(0, ap) \sim (0,p)$ and we have the $[(0,p)]$ associate class.

Now $F$ is a field and therefore $(b) = F$ for all $b \in F \setminus \{0\}$. Since $(1) = F = (b)$, $1 \sim b$ for all $b \in F \setminus \{0\}$ and $0 \sim (0)$ and $(1,0) \sim (b,0)$ and we will call this the $[(1,0)]$ associate class.

Finally, as above, $(ap) = (p)$ for $a \in U(\mathbb{Z}_{p^2})$ and thus $ap \sim p$ and $1 \sim b$ for all $b \in F \setminus \{0\}$, and therefore $(1, p) \sim (b, ap)$. We will call this the $[(1,p)]$ associate class.

Observe that 1 is not an associate to 0, 1 is not an associate to $p$, and 0 is not an associate to $p$. Therefore $V(\Gamma'(R_1)) = \{(0,1), (1,0), (0,p), (1,p)\}$.

Now, $(0,1)(1,0) = (0,0) \in R_1$, $(1,0)(0,p) = (0,0) \in R_1$, $(0,p)(1,p) = (0,0) \in R_1$ and therefore $\Gamma'(R_1) = [(0,1)] - [(1,0)] - [(0,p)] - [(1,p)]$.

Consider $R_2$. Let $r, s \in U(\mathbb{F}_q[x]/(x^2))$, and observe that $r \sim s$ because $(r)$ and $(s)$ both generate $\mathbb{F}_q[x]/(x^2)$. Observe that $0 \sim 0$ and $(0,1) \sim (0,r)$ and we will call this the $[(1,0)]$ associate class.

Observe that $\mathbb{F}_q[x]/(x^2)$ is a local ring and therefore has a maximal ideal which is $(x) = \{0, x, 2x, \ldots, (q-1)x\}$. Because $x$ is a unit, it will equal the ideal generated by any other unit in $R_2$. Thus $(ax) = (x)$ for $a \in U(\mathbb{F}_q[x]/(x^2))$, and thus $ax \sim (0,x)$ and in $R_1$, $(0,ax) \sim (0,x)$ and we will call this the $[(0,x)]$ associate class.

Now $F$ is a field and therefore $(b) = F$ for all $b \in F \setminus \{0\}$. Since $(1) = F = (b)$, $1 \sim b$ for all $b \in F$ and $0 \sim 0$ and $(1,0) \sim (b,0)$ and we will call this the $[(1,0)]$ associate class.
Finally, as above, \((ax) = (x)\) for \(a \in U(\mathbb{F}_q[x]/(x^2))\) and thus \(ax \sim (x)\) and \(1 \sim (x)\) for all \(b \in \mathbb{F}_q\), and therefore \((1, x) \sim (b, ax)\) and we will call this the \([1, x]\) associate class.

Observe that 1 is not an associate to 0, 1 is not an associate to \(x\), and 0 is not an associate to \(x\). Therefore \(V(\Gamma'(R_2)) = \{(0, 1), (1, 0), (0, x), (1, x)\}\).

Now, \((0, 1)(1, 0) = (0, 0) \in R_2\), \((1, 0)(0, x) = (0, 0) \in R_2\), \((0, x)(1, x) = (0, 0) \in R_2\) and therefore \(\Gamma'(R_2) = [(0, 1)] - [(1, 0)] - [(0, x)] - [(1, x)]\).

The following theorem generalizes Theorem 4.1 to the direct product of any two arbitrary infinite fields. The resulting associate class graph is complete and not self-looped.

**Theorem 4.2.** Let \(F_1\) and \(F_2\) be fields. Then \(\Gamma'(F_1 \times F_2)\) is \(K^2\) and neither vertex is looped.

**Proof.** Consider \(F_1 \times F_2\). Let \(m, n \in U(F_2)\), and observe that \(m \sim n\) because \((m)\) and \((n)\) both generate \(F_2\). Observe that \(0 \sim 0\) and \(m \sim 1\) in \(F_1\) and \(F_2\). Thus \((0, 1) \sim (0, m)\) and we will call this the \([(0, 1)]\) associate class.

Now \(F_1\) is a field and therefore \((b) = F_1\) for all \(b \in F_1\). Since \((1) = F_1 = (b)\), \(1 \sim b\) for all \(b \in F_1\), and note that \(0 \sim 0\). Thus, from the argument above, \((1, 0) \sim (b, 0)\) and we will call this the \([(1, 0)]\) associate class.

Observe that 1 is not an associate to 0. Therefore \(V(\Gamma'(R_1)) = \{(0, 1), (1, 0)\}\).

Now, \((0, 1)(1, 0) = (0, 0) \in F_1 \times F_2\), therefore \(\Gamma'(F_1 \times F_2)\) is \(K^2\) and neither vertex is looped.

It is important to note that this theorem not only categorizes the associate class graphs of the direct product of finite fields but also of infinite fields. It also does not restrict the direct product of the two fields to be of the same size, i.e., infinite or finite.

5 Understanding Ideals in \(R\) from \(\Gamma'(R)\)

Although our original conjecture was incorrect, we discovered that \(\Gamma'(R)\) does give us information about the ring from which it was created. Following [2, Theorem 2.3], listed below, we will show that the zero divisor graph being star shaped reducible, the associate class graph being star shaped reducible, and the zero divisors forming an ideal in \(R\), are all equivalent statements. The result is of importance because it shows that \(\Gamma'(R)\) is relaying information about the ring.

**Theorem 5.1.** Let \(R\) be a finite commutative ring with identity. Then the following are equivalent.

1. \(Z(R)\) is an ideal.
2. \(Z(R)\) is a maximal ideal.
3. \(R\) is local.
(4) Every $x \in Z(R)$ is nilpotent.

(5) There exists $b \in Z(R)$ such that $bZ(R) = 0$, and hence $\Gamma(R)$ is star shape reducible.

Utilizing (5) from Theorem 5.1, we were able to show that the associate class graph is star shape reducible.

**Theorem 5.2.** Let $R$ be a finite commutative ring with identity. Then the following are equivalent.

(1) $\Gamma(R)$ is star shape reducible.

(2) $\Gamma'(R)$ is star shape reducible.

(3) $Z(R)$ is an ideal in $R$.

**Proof.** (1) $\Leftrightarrow$ (3) is via [2, Theorem 2.3].

We now show (1) $\Leftrightarrow$ (2).

($\Rightarrow$) There exists $a \in V(\Gamma(R))$ that is connected to all other vertices and is self looped. Thus, for every $x \in Z(R)$, $xa = 0$. Consider $[a]_{assc} \in Z'(R)$. For every $[x]_{assc} \in Z'(R)\{[0]\}$, $[a]_{assc}[x]_{assc} = [0]_{assc}$ by definition of edge creation in $\Gamma'(R)$.

($\Leftarrow$) There exists $[a]_{assc} \in Z'(R)\{[0]\}$ where $[a]_{assc}[x]_{assc} = [0]_{assc}$ for all $[x]_{assc} \in Z'(R)$. Consider $a \in V(\Gamma(R))$. For every $x \in Z(R)\{[0]\}$, $ax = 0$ because $[a]_{assc} - [x]_{assc}$ if and only if $ax = 0$. And $[a]_{assc} - [0]_{assc}$ since $[a]_{assc}$ is the center of $\Gamma'(R)$. □

Listed below are future directions in which research in this area can move. Proving this conjecture will allow information about a ring to be extracted based on the associate class graph of the ring.

**Conjecture 1.** If $\Gamma'(R)$ is $K^2$ with neither vertex looped, then $R = F_1 \times F_2$.

**References**


