Validity, Truth and Faith in Mathematics

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I. Introduction

It is my goal here to write about mathematics and theology. Before I enter into a discussion of certain interconnections between these two fields, I should reveal what I think about mathematics itself so that the reader can understand the perspective from which I write. I am fully ready to admit that mathematics has a nature quite separate from what I may think of it. Theology also exists as a discipline and not simply as a collection of personal insights or opinions. The difference between these two areas may be that theology addresses, at least in part, something that is quite personal — a person's faith.

I would argue that when I discuss mathematics, I will not simply be objectively reporting on its nature. In some fundamental way I can not really say exactly what mathematics is, I have to report what I think it is. At least in the case of mathematics there is widespread agreement about its nature. Thus, what I write can be tested against the insights and opinions of others. Finally, however, our own view of the world is really quite personal, in its conception and in our expressions of reality as we see it. We need to be careful here; the reality about which physicists speak is something most of us believe has a separate, real, material nature. Mathematics is an abstract structure. It exists not in material objects, but in logic. When we use mathematics in application, we go beyond mathematics itself. While it may be possible to speak with confidence about this theorem, that definition and even a certain proof, to know mathematics is not to be able simply to recite a list of statements. Much of what follows is about mathematics in application, not mathematics itself. I write, as I must, in a very personal way about this subject.

We should all, of course, continually check with others to see that our views are not terribly out of step with others who write and speak with authority about our field. On the other hand, there are times when we must develop some confidence in our own thoughts and beliefs.

As a teacher, I want to do more than transfer information ("facts") to the students. I want to share enough of my insights (and even beliefs) so that they may be inspired to form their own "web of understanding." It will not be enough for them to recite certain mathematical rules, forms and procedures. They must come to their own deeper understanding so that they can continue to learn and to apply what they know. It is in this personal approach to teaching and understanding that I try to lead students to a view of mathematics which will help them make connections to many other disciplines. We might even go so far as to ask them to contemplate matters theological. To set the stage for that discussion, I first address the (for me) easiest part of this paper.

II. Validity as Relative Truth

While we casually speak of truth in mathematics, here we should be careful with such terms. Not only will different people use the term “truth” in different ways, one individual will use the same term to mean different things at different times and in different ways. Our consideration of this concept must begin with an overall look at logic as used in mathematics. In order to do this efficiently, I will use geometry as the mathematical topic of study. By less than a happy coincidence, geometry is my main area of study, but it also has a high place in the history of not only mathematics but of metamathematics.

While one could look into the early history of deduction itself and of ancient mathematics, here I will begin with the contributions of Euclid. We know very little about him except for his mathematical writings, but it is exactly those writings which are so important for the history and even nature of our discipline. Euclid lived around 350 B.C. in Alexandria and founded what we would today call the “chair in mathematics at the University of Alexandria.” He wrote the world’s most successful textbook, the Elements of Geometry. The Elements was a compilation of the most important mathematical “facts” available at the time. But it is so much more than that. Euclid’s work is the first of Greek mathematical writings to survive in its entirety, and since he compiled his work from mathematics existing at the time, it is hard to tell how much is original with him and how much is due to the work of others. The work is impressive however, not just for the content but for the way in which it is organized and presented.

We do know that the prestige of the Elements was so great in the ancient world that Euclid was referred to as “The Writer of the Elements” or simply, “The Geometer.”

We have to consider the work of Euclid as having been raised to the level of a paradigm. By “work” we mean both what he did and how he did it. Beyond just presenting the known mathematical knowledge of his time, Euclid organized the material as a chain of theorems (propositions) following the laws of deductive logic. The material moves from a few simple definitions and assumptions to propositions each of which is justified by something earlier. It was in doing this that Euclid firmly established what we know as the axiomatic method. Euclid unified a collection of isolated discoveries into a single deductive system, an axiomatic system, based upon a set of initial definitions and axioms. In such an elegant manner, he established geometry as a deductive “science.” Today, most of mathematics is constructed according to this same system. A theorem or proposition which we might call “true” is really one found to be logically consistent with the axioms and earlier established theorems. In other words, there is a valid argument or proof supporting this claim to truth. More about this later.

In laying the foundation of his geometry, Euclid began with 23 definitions. Over the years, mathematicians have scrutinized the Elements and reached the conclusion that Euclid defined too many terms and in that way created some definitions which were not useful. For example, in one translation of the Elements, his first definition is, “a point is that which has no part.” Another translation of this is, “a point has position, but no magnitude.” Neither definition is very useful in particular because we need to know the meaning of “part” or “magnitude.”

The nature of deduction will never allow for all definitions to be completely stated in a non-circular manner in terms of earlier defined terms. We should insist that when a definition uses undefined terms, these should be at least as clearly understood as the term being defined.

The majority of Euclid's definitions were quite satisfactory. From the difficulties that Euclid did have, it became quite evident that to be successful with the axiomatic method, a sufficient number of terms must be left undefined. While these terms are not defined in an axiom system, it is hoped that their sense will be "intuitively" clear. Over the years other axiom systems have been devised to serve as a logical basis for Euclidean geometry. Perhaps the most respected system was created by David Hilbert (1862–1943) in 1902. It is important to know that Hilbert leaves such terms as "point," "line," and "plane" undefined. While not defined, these terms are related to others as they are used in axioms or postulates.

After the definitions, Euclid recognized the need for beginning assumptions, statements to be accepted without proof. Today such statements are called axioms or postulates. It appears that Euclid used the term "postulate" for assumptions which pertain only to geometry. Statements applicable to all areas of mathematics were what he called axioms or common notions. The early view was that such assumptions were so clearly true that they were beyond question. Embedded in this statement is an assumption that the nature of truth itself is clearly understood. Some have called axioms "God-given truths" or "self-evident truths." Whatever we might want to say about the "truth" of axioms, it is critical to note that this is not at all like the "truth" of a theorem. The theorem is proved to be logically consistent in the system. It might be better here to call a theorem a "valid" result. To say that an axiom is "true" is to go beyond the logical system. We are not claiming that an axiom is proved in any way. Let us set aside this major question for a bit. Euclid's axioms (common notions) were:

1. Things which are equal to the same thing are equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

One might argue that these statements do indeed appear to be "true." That is, they say something which always corresponds to the way things are when these statements make sense about the world. But are they about the world? The fifth common notion gives us some insight as to how axioms may be developed. It is probable that Euclid was dependent on the physical world for his axioms and postulates. If we consider a finite collection of objects, then it is clear that the whole is more than the part. But Euclid does not intend these common notions to be about so many cows in the field. The objects in his geometry have an abstract nature, and it is clear that there are an infinite number of things to be considered. There are mathematically reasonable ways to define "greater" in reference to infinite sets, and we cannot say that "The whole is greater than the part" will be accurate in this context. This is just one problem with a claim that axioms represent some kind of "universal truth."

Besides the common notions, the 465 theorems of Euclid's Elements were also based upon the following postulates:

(1) A straight line can be drawn from any point to any point.
(2) A finite straight line can be produced continuously into a line.
(3) A circle can be described with any center and any distance.
(4) All right angles are equal to one another.
(5) If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, then the two straight lines (if produced indefinitely), meet on that side on which the angles are less than two right angles.

Postulates (1) and (3) are statements of existence. The first postulate allows us to conclude that given any two different points, there must be a straight line determined by these points. We sometimes say that "two points determine a straight line." The third postulate gives the existence of a circle which Euclid describes in his fifteenth definition.

The first three postulates really fail to say what Euclid exactly intended. Over the years, mathematicians have closely studied the proofs in the Elements to check to see that all statements of reasoning were based on theorems already proved along with the axioms and postulates. This is necessary to assure that the mathematics follows the axiomatic method. In these studies, it became evident that Euclid omitted establishing uniqueness in these three postulates. It is clear from the use in his proofs that this is what he intended but did not make explicit.

Such omissions or tacit assumptions of Euclid were, over time, pointed out and corrected by those mathematicians who have attempted to put Euclidean geometry on a firm foundation by producing an axiom system free of flaws.

It is important for us to consider Euclid's Fifth Postulate, one of the most famous passages in mathematics. Because it was stated more like a theorem (notice the use of if, then) and because it was not a simple, concise affirmation, over the centuries mathematicians felt that it should not be a postulate but should be proved from the list of postulates and axioms which comes before it. It would be best to have an axiom system which is independent. This would be the case if no single axiom could be logically deduced from the other axioms. An independent axiom system would be more efficient and would not run the risk of containing internal inconsistencies.

From the earliest days to the last century, it was thought that the Fifth Postulate was not independent of the other postulates and axioms. It is quite possible that Euclid himself may have doubted the independence of the Fifth Postulate, because it is clear that he delayed using this postulate until the last possible moment. Even so, Euclid did work at finding a statement for his axiom that could be thought of as self evident. So we must know that Euclid and those who followed in the next two thousand years didn't really doubt the "truth" of the Fifth Postulate; they just wanted to see its "truth" established by a proof rather than having to be assumed to be "true." But the history of geometry shows that all such attempts to prove the statement failed. From time to time a "proof" would be announced only to be discovered invalid because of a flaw in the logic or because something was used which was equivalent to the Fifth Postulate.

Many well-known results in Euclidean geometry depend upon the fifth postulate (or something equivalent to it).

Below are a few such results which one might recall from a high school course in geometry.

In any triangle, the three interior angles are equal to two right angles (180°).

Parallel lines are everywhere equidistant. For a right triangle, the square on the hypotenuse is the sum of the squares on the two sides.

This last result is known to us as the Pythagorean Theorem and was considered so important that Euclid proved it twice. We can show logically that these results cannot be proved unless we accept the Fifth Postulate. So these results are not "absolute truths" in geometry, even though they may be so considered by high school students (and even some high school teachers). That is, it is not the case that these results are consistent in any mathematical system which may be called a geometry. These statements are logically equivalent to the Fifth Postulate; to accept any one is to be able to then prove any other.

For over two thousand years, there was only one geometry, and it was held up as a remarkable triumph of logic. It was viewed as not merely valid, but a truth of the universe. For example, we can see in the writings of Immanuel Kant that Euclidean geometry was given the status of revealing the "true" nature of space. That space, as described by the axiom and postulates, was a necessary "truth" about the material world. As we said before, this is going beyond mathematics itself.

By the middle of the nineteenth century, the mathematical community finally agreed that Euclid had been correct all along. Not only had all attempts to show dependence of the Fifth Postulate failed, but there was developed a new system (a new geometry) which proved to be as valid as the old geometry.

It is instructive to see, at least in a general way, how this happened. An axiom system is consistent if no contradiction will follow from the basic assumptions. In other words, if a system of axioms is such that we can prove a theorem which contradicts a previously known theorem or previously stated axiom, then the system is not consistent. When some mathematicians attempted to prove the dependence of the Fifth Postulate, they did so by replacing it with a statement contradictory to the Fifth Postulate. The resulting system of axioms and postulates was then thought to be inconsistent. If a contradiction could have been proved, it would have been proper to conclude that the Fifth Postulate was, in fact, dependent on the other assumptions. No contradictions were found, but in the search for such contradictions, some axioms systems were created which were themselves as independent as is Euclidean geometry (this could be proved) and which were decidedly non-Euclidean. First came what we call hyperbolic geometry (attributed to Bolyai and Lobachevski). Other geometries were to come later.

With the acceptance of these other geometries, Euclidean geometry lost its place as the geometry of the universe. About the same time the paradigm of Newtonian physics was beginning to fall apart. The special place of Euclidean geometry had disappeared, and what had been the universal geometry now became simply Euclidean geometry. One consequence was that an axiom system did not have a claim as a "self-evident, universal truth." We would want our axiom system to be consistent (free from contradiction). It turns out that this is not so simple to show, but more about that later. It might be worth mentioning that the nineteenth century was a time in which mathematical rigor became a central concern in general. Much time was spent in providing a rigorous logical foundation for the calculus even though it had been so successfully applied in physics. On second thought, we might argue that because calculus had been so widely applied, it was important to place it on a firm foundation. We should also note that this calculus of Newton and Leibniz was an analysis which took place in a Euclidean space.

III. Faith in Axioms

We began this discussion in an attempt to locate validity and truth in mathematics, if possible. The examination of geometry above should demonstrate that while we may have validity, we obtain only "relative truths." Some of our favorite theorems in geometry and those which may be held up as examples of "truth" are seen to be only valid (in the sense that the result follows from earlier results or axioms using deductive logic) within one particular geometry. It is not within mathematics to even make sense of the question, "Which is the real geometry of the universe?" In hyperbolic geometry, triangles do not have an interior angle sum of 180 degrees, and parallel lines are not everywhere equidistant. While you may prefer one geometry to another, both Euclidean and hyperbolic geometries are equally consistent. An early faith in the ultimate "truth" of the axioms has been replaced by a (perhaps sad) knowledge that one set of axioms is really just different from another but not logically superior. A colleague in philosophy has encouraged me to regard, say, the second common notion (If equals be added to equals, the wholes are equal) as "really true." Since we might view mathematics as a collection of syllogisms, it would be nice to know that the initial premises are "really true," but it is not a useful mathematical exercise since we are concerned about the structure created, and we know equally valid but contradictory structures may be created. We may all want to "accept" common notion 2, but what do we do with the Fifth Postulate? There are still many people who see Euclidean geometry as the only "true" geometry (and many others who think it is the only geometry). To use the word "true" in this sense is, I would claim, to misunderstand or confuse what mathematics is and how it can be used.

So if we don't acknowledge Euclid's axioms as "truth," what is our foundational belief? As we start a formal development of mathematics, we accept the "rules" of deductive logic itself. We say that those are the rules by which we shall play. So it may be correct to say that we, as mathematicians, have a "faith" in deductive logic. Later I will want to argue that we have, in a similar way, a "faith" in the consistency of our axiom system for geometry (or algebra, etc.).

But I am not claiming that this kind of belief is theological in nature. The goal so far has been to demonstrate that mathematics, regardless of the kind of image it may have, is not about "truth" in any external or absolute sense. If I give you two cows and you already have four cows, you will then have six cows. On the other hand, if you mix a pint of water and a pint of salt, you do not get two pints of mixture. Neither of these examples of "applied mathematics" tells us anything about the validity of the algebra of the set of natural numbers. This is quite an important point. We cannot make any claim about the ultimate "truth" of mathematics by looking at the power of its application. In the end, mathematics is not about the real world at all, but more about this below.

IV. The "Miracles" of Math

In order to move to a discussion of matters theological, we will have to ask some questions beyond those about the internal structure of mathematics. If we concentrate on the modern view of mathematics as a collection of various axiom systems, some contradictory to others, what are we to make of this strange mixture? Are axioms truly arbitrary (except for the demands of independence and consistency)? This may well be the message of the history of mathematics as seen in the development of non-Euclidean geometries alone. But then, how do we account for the power of mathematics? Physicist Eugene Wigner wrote a famous article called "The Unreasonable Effectiveness of Mathematics in the Natural Sciences."6 We are reminded over and over that mathematics has been (and continues to be) surprisingly effective in application. The digital computer has become commonplace to a young generation, and yet its development was not possible without the insights of sophisticated mathematics. Of course, we should also acknowledge the quite sophisticated physics and engineering which have gone into the computer, but those fields owe an essential debt to mathematics also. So even though mathematics can help us bring about these impressive results, we cannot simply look inside mathematics, at its internal structure, and see why it is so effective.

Also, while mathematics has been found to have many wonderful and profound ways that it may be applied, there is another aspect of the subject which needs to be examined. In addition to being surprised by the power of mathematics, many of us are surprised by some of the "mysteries" of mathematics. For example, we know (and Euclid knew) that there are an infinite number of prime numbers (whole numbers divisible evenly only by themselves and one). But there are many unanswered questions about primes. For example, we don't know if the list of "twin primes" is unending. These are pair of primes, two units apart, such as 11 and 13 or 29 and 31. Not all primes are paired (37 is prime but 35 and 39 are not), but as far as we can compute, we keep finding another "twinned" pair sooner or later. We don't know much at all about the ways primes are distributed among the whole numbers, but we do know that important theorems in many fields of mathematics show special properties for primes. Perhaps some day we will have an answer to the question about the possible limit on the number of twin primes (I doubt it), but the question about the nature of their distribution is possibly not a mathematical question at all.

V. So What?

What are we to make of a subject which has proved so remarkably effective in application and yet appears so arbitrary in content? It's hard not to confront this question if one undertakes any real contemplation of mathematics. We do need to accept that mathematics is, in fact, impressive in application, but I should explain why I characterize the content of mathematics as "arbitrary." If one axiom system is as "good" as another, and if we may select our axioms at will (avoiding contradictions), then isn't any particular mathematics we study really just our arbitrary creation? Why would such a structure be so effective in application? Two possible answers should be considered.

First, we might argue that even though we have choice in selection of axioms, what we really do is select axioms which are abstraction from experience. In this way, perhaps we could view Euclidean geometry as the obvious geometry of our surroundings. Common experiences shape our intuition and this leads to our formation of axioms. Following this line of argument, can it be a surprise if the mathematics we then develop ends up being valuable in application? We should be cautious about taking this line of argument too far. On the one hand, this thinking might explain the development of a very powerful system like Newtonian physics (often hailed as a true insight into God's design for the universe). On the other hand, how are we to understand that, early in this century, relativity theory and non-Euclidean geometry gave us astronomical predictions which proved more accurate than those predicted by Newtonian physics? Kuhn has written of scientific revolutions as showing us that science does not present us with truth itself but simply with a model that, however good it may seem to be at any one time, may someday be replaced by another model.4

But perhaps we should look not to the content of mathematics but to its form in order to explain its power in application. This second possible explanation might emphasize the pure logical form of mathematics as so perfect that reasoning takes place therein in such a way as to provide for the best results given a certain set of assumptions at the outset.

The history of modern philosophy begins with Descartes, who had an agenda to illuminate "the whole of science, even the whole of knowledge, by one and the same method: the method of reason." [2, p.4] His famous, "Discourse on the Method of Properly guiding the Reason in the Search of Truth in the Sciences," is a most important work in the history of philosophy. But, "Descartes was first and foremost a geometer; he claimed he was in the habit of turning all problems into geometry. What gives the method substance is the use of mathematics, the science of space and quantity ...." [2, p. 5] This approach is perhaps easy to understand knowing that Descartes worked in a day when all were certain that geometry revealed absolute truth. But to be fair to Descartes, the method was the focus.

VI. The Fall

I grew up with a nice, comforting image of mathematics as a wonderfully pure and perfect system. In my undergraduate career, I did come to know about non-Euclidean geometries, but I was never exposed to some real problems with axiom systems that had been revealed in the 1930s. While mathematics was clearly expanding, and old questions would, from time to time, be answered, it turns out that some questions could never be settled. In 1932, Kurt Godel proved that an axiom system of the scope that would at least support ordinary arithmetic will always admit of statements which cannot be proved "true" or "false."3 We can take such a statement and add it to the axiom system to obtain a new system, but it will also be similarly incomplete. That continues forever. It is a logical consequence of this that one cannot prove that such an axiom system is truly consistent — free of contradiction. We can study a system for years (decades, centuries, ...) and never find a contradiction, but we will still not have the (logical) comfort of knowing the system is really "pure." Our favorite geometry, which we had come to understand was not unique, now might even be inconsistent. I was certainly shocked to learn of this "fall from grace," but in spite of such knowledge, we labor on with a faith that we are working with something valuable.

VII. The Clockmaker

Hans Kung writes that, "The question of the ultimate foundations and the ultimate meaning of mathematics, then, remains open today." He goes on to point out that work goes on, "in both pure and applied mathematics — with impressive successes. Natural science and technology live on it,...[but] the universal claims of mathematical — scientific thought have been shaken to their foundations." [5, p.33] Next, writing about the possibility of conclusive proof of God, Kung notes that, "Evidently it is easier to reach certainty of the self from a certainty of God presupposed by faith than conversely to gain certainty of God from a philosophically proved certainty of self." [5, p.34]

But how can we come to belief? In a section where he examines the work of Pascal, Kung writes that, "The credibility of faith\/, however, is also something that cannot be assumed. Neither can we dispense with reason and rationality and build up faith merely on authority..." [5, p.60] One is led to a conclusion that, "Even in faith, then, there is no certainty entirely free from doubt. In faith, we must commit ourselves to something uncertain." [5, p. 61] Finally, "For Pascal, however, in the last resort, reason is not the basis of faith, but faith is the basis of reason." [5, p. 63]

I find this a persuasive argument coming from a mathematician who clearly understood the geometry of his time and who today would be, in many ways, not that disappointed by the revelations of contemporary mathematics. In the old view, geometry was discovered. People believed that geometry revealed absolute truths of the universe, and such a grand, intricate design certainly could lead one to contemplate the designer.

If we came upon a clock, we might believe that there was, somewhere, a clockmaker. In fact, I argue that we would be certain that there had been a clockmaker. This is the kind of "axiom" in which we could have real faith. Of course, the argument from design for the existence of God is not simply an analogy to the clockmaker. For one thing, a clock is an artifact we recognize. People like us make clocks. It is a bit disingenuous to ask us to imagine that we had never seen a clock and that we came upon one. Also we are separate from the clock and rather like the clockmaker, whereas in the design of the universe, we are part of the design. Still I find a kind of simple, direct appeal to the argument from design. Whether we consider subatomic particles, complex molecules or the structure of the cell, we should be overwhelmed by the delicate and intricate design. But the sciences which have revealed these things to us can be used to provide explanations for existence which leave out a designer. And when religion sought to deny scientific insight, we were all invited to divide our thought into two separate categories. We could achieve some degree of peaceful coexistence by isolating the other kind of thinking.

But today, we don't have a mathematics which represents pure, perfect reason. We don't have a science which represents "absolute truth." We don't have a certain glimpse into the actual design of the universe; we have models; we made the models. Some of them are quite intricate themselves, and they are, in places, somewhat mysterious. Some of these models seem to explain our observations more effectively than others. They give us hope that we can continue to learn more while always knowing that what we learn is contingent and based, in part, on some articles of faith. I am persuaded that it is exactly this lack of perfection that renders us humble before the mysteries of the universe and gives us the courage to embrace the argument by design as believable not because of the power of logic but because of the power of faith.

Notes:

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